

Design of LDPC Codes Robust to Noisy Message-Passing Decoding

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Abstract—We address noisy message-passing decoding of low-density parity-check (LDPC) codes over additive white Gaussian noise channels. Message-passing decoders in which certain processing units iteratively exchange messages are common for decoding LDPC codes. The exchanged messages are in general subject to internal noise in hardware implementation of these decoders. We model the internal decoder noise as additive white Gaussian noise (AWGN) degrading exchanged messages. Using Gaussian approximation of the exchanged messages, we perform a two-dimensional density evolution analysis for the noisy LDPC decoder. This makes it possible to track both the mean, and the variance of the exchanged message densities, and hence, to quantify the threshold of the LDPC code in the presence of internal decoder noise. The numerical and simulation results are presented that quantify the performance loss due to the internal decoder noise. To partially compensate this performance loss, we propose a simple method, based on EXIT chart analysis, to design robust irregular LDPC codes. The simulation results indicate that the designed codes can indeed compensate part of the performance loss due to the internal decoder noise.

I. INTRODUCTION

LOW-density parity-check (LDPC) codes – first discovered by Gallager [1] and rediscovered by Spielman *et al.* [2] and MacKay *et al.* [3], [4] – due to their outstanding performance have attracted much interest and have been studied extensively during the recent years. They have also been included in several wireless communication standards, e.g., DVB-S2 and IEEE 802.16e [5], [6]. Effective decoding of LDPC codes can be accomplished by using iterative message-passing schemes such as belief-propagation algorithm or sum product algorithm [7]. Moreover, powerful analytical tools including *EXIT chart analysis* [8], [9] and *density evolution analysis* [10] are developed for designing LDPC codes and quantifying their performance limits.

The sum-product algorithm is based on elementary computations using sum-product modules [11]. These modules can be implemented using digital circuits or analog circuits. The digital implementation of sum-product modules for LDPC decoding, e.g., the one presented in [12], is subject to noisy message passing due to the quantization of messages. The impact of quantization error on LDPC decoding performance

is investigated in [13], where the simulations indicate the resulting substantial performance degradation. Quantization error is often modeled as an additive noise and is assumed Gaussian under certain conditions [14]. In [15], Varshney considers the impact of quantization error on message passing algorithm, and suggests a signal-independent additive truncated Gaussian noise model. More recently, Leduc-Primeau *et al.* studied the impact of timing deviations (faults) on digital LDPC decoder circuits in [16]. They model the deviations as additive noise to the messages passed in the decoder, and show that under certain conditions the noise can be assumed Gaussian.

Loeliger *et al.* in [11] and Hagenauer and Winklhofer in [17] introduced soft gates in order to implement sum-product modules using analog transistor circuits. They have also shown that by the variations of a single basic circuit, the entire family of sum-product modules can be implemented. Using these circuits, any network of sum-product modules, in particular, iterative decoder of LDPC codes can be directly implemented in analog very-large-scale integrated (VLSI) circuits. In analog decoders, the exchanged messages are in general subject to additive intrinsic noise. The power of the noise depends in part on the chip temperature [18]. To capture this phenomenon, in [18] Koch *et al.* considered a channel that is subject to an additive white Gaussian noise. This channel is motivated by point-to-point communication between two terminals that are embedded in the same chip. Therefore, the *internal decoder noise* may affect the communication of soft gates, and hence degrades the performance of the iterative analog decoder.

Since in practice digital or analog LDPC decoders are subject to internal noise, the impact of this noise on the performance of iterative decoding needs to be investigated. Performance analysis of noisy LDPC decoding has attracted extensive interest recently (see e.g. [15], [19]–[23] and the references therein). The performance of a noisy bit-flipping LDPC decoder over a binary symmetric channel (BSC) is studied in [15]. In this setting, the decoder messages are exchanged over binary symmetric internal channels between the variable nodes and the check nodes. It has been shown that the performance degrades smoothly as the decoder noise probability increases. Tabatabaei *et al.* studied the performance limits of LDPC decoder when it is subject to transient processor error [20], [21]. This research was further generalized in [22] by considering both transient processor errors and permanent memory errors, using density evolution analysis for regular LDPC codes.

In this paper, we analyze the performance of LDPC codes

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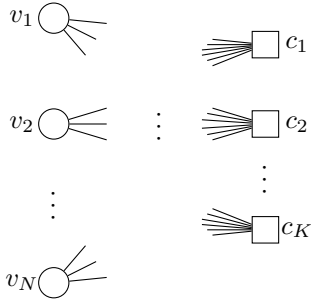


Fig. 1. Tanner graph of a regular (3, 6) LDPC code, where squares denote check nodes and circles denote variable nodes.

transmitted over an AWGN communication channel, when a sum-product decoding algorithm is employed in which exchanged messages are degraded by independent additive white Gaussian noise. We first invoke a density evolution (DE) analysis to track the probability distribution of exchanged messages during decoding, and quantify the performance degradation due to the decoder noise. We compute the density evolution equations for both regular and irregular LDPC codes. Also, we introduce an algorithm to find the EXIT curves of a noisy decoder. Finally we propose an algorithm for the design of robust irregular LDPC codes using EXIT chart for noisy decoders to partially compensate the performance loss due to the internal decoder noise.

This paper is organized as follows. In Section II, we present the definitions and model for noisy message-passing decoder. In Section III, we derive the density evolution equations for the noisy message-passing decoder. Next, numerical results of the density evolution analysis and simulation results of finite-length codes are presented. In Section IV, EXIT chart analysis of the noisy decoder is presented. Using the EXIT charts, a method for designing robust codes for the noisy decoder is presented in Section V. Finally, Section VI concludes the paper.

II. LDPC CODES AND NOISY MESSAGE-PASSING DECODING PRINCIPLES

Consider a regular binary (d_v, d_c) LDPC code with length N . The code can be represented by a $K \times N$ parity check matrix \mathbf{H} with binary elements, where the weight of each column and row of the matrix are d_v and d_c , respectively. There is a Tanner graph corresponding to the parity check matrix with N variable nodes and $K \triangleq N \frac{d_v}{d_c}$ check nodes. Every variable node in the graph is connected to d_v check nodes and every check node is connected to d_c variable nodes. Corresponding to the ones in the columns and the rows of \mathbf{H} , the variable nodes and the check nodes are connected to each other in the Tanner graph. Fig. 1 exemplifies a Tanner graph for a regular (3, 6) LDPC code with length N . Variable node v_i and check node c_j are known as *neighbors*, if they are connected to each other.

Message-passing decoding algorithm can be represented as iterative exchange of messages between check nodes and variable nodes of the Tanner graph. Specifically, every check node receives messages from its d_c neighbor variable nodes and

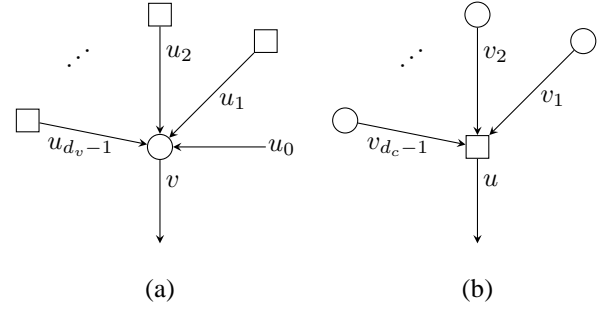


Fig. 2. Message flow through a variable node (a), and through a check node (b).

sends the computed messages back. Similarly, each variable node exchanges messages with its d_v neighbor check nodes.

We consider the output messages of the variable and check nodes as log-likelihood ratio (LLR) values, where the sign of a variable node message specifies the bit estimate and its magnitude indicates the reliability of the estimation.

According to the sum-product decoding algorithm, the message at iteration l from a variable node to a check node, denoted by $v^{(l)}$, is

$$v^{(l)} = \sum_{i=0}^{d_v-1} u_i^{(l-1)}, \quad (1)$$

where $u_i^{(l-1)}$, $i = 1, \dots, d_v - 1$, are incoming LLRs from variable node neighbors at iteration $l - 1$, except the check node that is to receive the output message $v^{(l)}$, and u_0 is the incoming LLR message from the communication channel. The message $u^{(l)}$ from a check node to a variable node at iteration l can be obtained as follows

$$\tanh \frac{u^{(l)}}{2} = \prod_{j=1}^{d_c-1} \tanh \frac{v_j^{(l)}}{2}. \quad (2)$$

Fig. 2 shows the schematics of message-passing for a variable node and a check node.

For a noisy decoder, the output messages of the variable and check nodes are subject to additive white Gaussian noise. The conventional model shown in Fig. 2 can be extended to the one in Fig. 3, where n_i and ν_j denote the additive white Gaussian noise affecting the output messages of check nodes and variable nodes, respectively. Hence, γ_j and μ_i are noisy versions of v_j and u_i , respectively. Therefore, the incoming messages to the variable nodes and the check nodes are

$$\mu_i^{(l)} = u_i^{(l)} + n_i, \quad (3)$$

$$\gamma_j^{(l)} = v_j^{(l)} + \nu_j, \quad (4)$$

where n_i and ν_j are assumed to be independent and identically distributed (i.i.d.), i.e., $n_i, \nu_j \sim \mathcal{N}(0, \sigma_d^2)$.

According to the sum-product algorithm, at iteration l the decoding is performed based on the following updating equations

$$v^{(l)} = u_0 + \sum_{i=1}^{d_v-1} \mu_i^{(l-1)}, \quad (5)$$

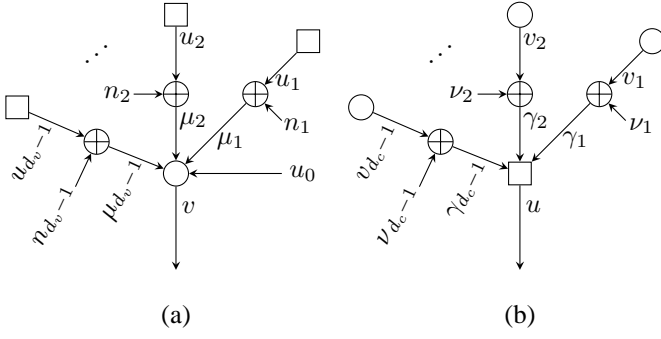


Fig. 3. The model of (a) a variable node (b) a check node in noisy belief-propagation decoder.

$$\tanh \frac{u^{(l)}}{2} = \prod_{j=1}^{d_c-1} \tanh \frac{\gamma_j^{(l)}}{2}. \quad (6)$$

In order to generalize these equations to irregular case one can follow the same steps as the one described in [7]. In the next section, we propose an approach for the performance analysis of this noisy message-passing decoding algorithm.

III. DENSITY EVOLUTION ANALYSIS OF NOISY DECODER

In this section, using Gaussian approximation, we will find the density evolution equations for the noisy decoder with regular variable and check degrees. We further generalize the results to the irregular case, and finally use the derived density evolution equations to evaluate the performance of noisy decoders.

A. Gaussian Approximation and Consistency

The density evolution analysis is an analytical method for tracking the densities of messages in iterative decoders. This can be used to predict the performance limits of an LDPC code measured by code's threshold [24]. The code's threshold is the smallest (largest) communication channel SNR (noise variance) for which an arbitrarily small decoding bit-error probability can be achieved by sufficiently long codewords. For an AWGN communication channel and an LDPC sum-product decoder, the densities of the exchanged messages between the check nodes and the variable nodes can be approximated as Gaussian [7], [25]. Hence, these densities may be characterized only with their mean and variance. A Gaussian random variable whose variance is twice its mean is said to be *consistent* [15]. The consistency assumption simplifies density evolution as a one-dimensional recursive equation based on the mean (or the variance) of the messages. In [7], this assumption is used for the DE analysis of a *noiseless* LDPC decoder, and subsequently, quantifying the threshold of the code.

The key assumption in the density evolution analysis of noise-free decoders is that the code block length is sufficiently large, based on which it may be assumed that the Tanner graph of the LDPC code is cycle-free. Since the code is linear and the communication channel is symmetric, we consider the transmission of an all-one codeword using a binary phase

shift keying (BPSK) modulation. Thus, the LLR values received over an AWGN communication channel are Gaussian distributed. The mean and the variance of the received LLR values are respectively equal to $m_0 = \frac{2}{\sigma_n^2}$ and $\sigma_0^2 = \frac{4}{\sigma_n^2}$, where σ_n^2 is the channel noise variance [7]. We assume that the random variables u , v , u_i and v_j are all Gaussian distributed. First, we check whether the messages of a noisy sum-product LDPC decoder are consistent. To this end, we consider the expected values of both sides of (3) and (5) and obtain

$$m_v^{(l)} = m_0 + (d_v - 1)m_u^{(l-1)}, \quad (7)$$

where $m_v^{(l)}$ and $m_u^{(l-1)}$ denote the means of output messages of variable nodes and check nodes, respectively. The index i is omitted since u_i , $i = 1, \dots, d_v - 1$, are i.i.d. Next, by computing the variances of both sides of (3) we have

$$\sigma_{\mu_i}^{2(l)} = \sigma_{u_i}^{2(l)} + \sigma_d^2. \quad (8)$$

Using (5), we obtain the variance of the variable node output

$$\sigma_v^{2(l)} = \sigma_0^2 + \text{var}\left(\sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right) + 2\text{cov}\left(u_0, \sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right), \quad (9)$$

where $\text{var}(X)$ denotes the variance of random variable X , and $\text{cov}(X, Y)$ is the covariance of random variables X and Y . Since μ_i , $i = 1, \dots, d_v - 1$, are i.i.d. Gaussian random variables, we have

$$\text{var}\left(\sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right) = \sum_{i=1}^{d_v-1} \text{var}\left(\mu_i^{(l-1)}\right) = (d_v - 1)\sigma_{\mu}^{2(l-1)}. \quad (10)$$

The last term in (9) is zero, as the Tanner graph of the code is assumed to be cycle-free and u_0 is independent of the noisy messages. Therefore, the variance of a variable node output message at iteration l can be simplified as follows

$$\sigma_v^{2(l)} = \sigma_0^2 + (d_v - 1)\sigma_u^{2(l-1)} + (d_v - 1)\sigma_d^2. \quad (11)$$

To verify the consistency of the noisy decoder, we plug in $\sigma_v^{2(l)} = 2m_v^{(l)}$ and $\sigma_u^{2(l-1)} = 2m_u^{(l-1)}$ into (11) and compare it with (7). It is clear that as long as σ_d^2 is non-zero, the two quantities are not equal and hence the noisy LDPC decoder is *not consistent*. Therefore, it does not suffice to track only the mean values of the nodes' output messages. Instead, it is required to track both the mean and the variance of nodes' output messages. A similar situation has been shown in the simulation results of [26], when there is an incorrect estimation of the communication channel SNR at a (noiseless) LDPC decoder. However, since the code is linear and the communication channel is symmetric, sending all-one codeword is sufficient for statistical performance evaluation of the code [26].

For an irregular LDPC code, the degree distribution of variable nodes is $\lambda(x) = \sum_{i=2}^{D_v} \lambda_i x^{i-1}$ and that of the check nodes is $\rho(x) = \sum_{i=2}^{D_c} \rho_i x^{i-1}$, where λ_i and ρ_i are the percentage of edges that are connected to variable nodes and check nodes of degree i , respectively. D_v is the maximum degree of variable nodes and D_c is the maximum degree of

TABLE I
RELATION BETWEEN THE THRESHOLD AND DECODER NOISE VARIANCE

σ_d^2	SNR threshold SNR_{th}	$(\sigma_n)_{\text{th}}$
0	1.163 dB	0.8744
1	2.835 dB	0.7215
2	3.635 dB	0.6580
3	4.185 dB	0.6177

check nodes. In this case, by similar steps as the ones for the regular LDPC codes, it can be shown that for the mean and the variance of a variable node of degree i at iteration l

$$m_{v,i}^{(l)} = m_0 + (i-1)m_u^{(l-1)}, \quad (12)$$

$$\sigma_{v,i}^{2(l)} = \sigma_0^2 + (i-1)\sigma_u^{2(l-1)} + (i-1)\sigma_d^2, \quad (13)$$

where $m_{v,i}^{(l)}$ and $\sigma_{v,i}^{2(l)}$ are the mean and the variance of a variable node of degree i at iteration l , respectively. From (12) and (13) it can be inferred that the consistency is not valid for the irregular case.

B. Density Evolution with Gaussian Approximation for Noisy Message-Passing Decoder

In the case of a noisy LDPC decoder, we have shown that consistency does not hold and we should track both the mean and the variance of nodes' output messages. In order to do this, we use the key equations (3)-(8) and (11). By computing the expected value of both sides of equation (6), and noting that $\gamma_j^{(l)}$, $j = 1, \dots, d_c - 1$ are i.i.d., we have

$$\begin{aligned} \mathbb{E} \left[\tanh \frac{u^{(l)}}{2} \right] &= \mathbb{E} \left[\prod_{j=1}^{d_c-1} \tanh \frac{\gamma_j^{(l)}}{2} \right] \\ &= \left(\mathbb{E} \left[\tanh \frac{\gamma^{(l)}}{2} \right] \right)^{d_c-1}, \end{aligned} \quad (14)$$

where $\gamma^{(l)}$ is defined in (4) and has the following distribution

$$\gamma^{(l)} \sim \mathcal{N}(m_v^{(l)}, \sigma_v^{2(l)} + \sigma_d^2). \quad (15)$$

Next, by computing the expected value of squared tanh rule, we obtain the second major equation as follows

$$\mathbb{E} \left[\tanh^2 \left(\frac{u^{(l)}}{2} \right) \right] = \left(\mathbb{E} \left[\tanh^2 \left(\frac{\gamma^{(l)}}{2} \right) \right] \right)^{d_c-1}. \quad (16)$$

The density evolution can be obtained by simultaneously solving equations (14) and (16). Specifically, representing $\gamma^{(l)}$ using (15) and $m_v^{(l)}$, $\sigma_v^{2(l)}$ from (7) and (11), we obtain the DE equations for check nodes as follows

$$\begin{aligned} f(m_u^{(l)}, \sigma_u^{2(l)}) &= \left(f(m_v^{(l)}, \sigma_v^{2(l)} + \sigma_d^2) \right)^{d_c-1}, \\ g(m_u^{(l)}, \sigma_u^{2(l)}) &= \left(g(m_v^{(l)}, \sigma_v^{2(l)} + \sigma_d^2) \right)^{d_c-1}. \end{aligned} \quad (17)$$

The auxiliary functions $f(m, \sigma^2)$ and $g(m, \sigma^2)$ are defined as

follows

$$\begin{aligned} f(m, \sigma^2) &\triangleq \mathbb{E} \left[\tanh \left(\frac{X}{2} \right) \right], \\ g(m, \sigma^2) &\triangleq \mathbb{E} \left[\tanh^2 \left(\frac{X}{2} \right) \right], \end{aligned} \quad (18)$$

where $X \sim \mathcal{N}(m, \sigma^2)$. These equations can be used to track $m_u^{(l)}$ and $\sigma_u^{2(l)}$ in the decoding iterations of a regular (d_v, d_c) LDPC for given values of communication channel noise variance and internal decoder noise variance. Because of non-linearity of the equations in (17), it is not easy to find a closed form expression for the mean and the variance at each iteration. As such, we resort to Monte-Carlo simulations to solve (17) and adopt the semi-Gaussian approximation method used in [27] and [19].

Similarly, for irregular LDPC codes, the message distributions are approximated by Gaussian mixture [7]. For each check node of degree i at iteration l we have

$$\begin{aligned} f(m_{u,i}^{(l)}, \sigma_{u,i}^{2(l)}) &= \left[\sum_{j=2}^{D_v} \lambda_j f(m_{v,j}^{(l)}, \sigma_{v,j}^{2(l)} + \sigma_d^2) \right]^{i-1}, \\ g(m_{u,i}^{(l)}, \sigma_{u,i}^{2(l)}) &= \left[\sum_{j=2}^{D_v} \lambda_j g(m_{v,j}^{(l)}, \sigma_{v,j}^{2(l)} + \sigma_d^2) \right]^{i-1}, \end{aligned} \quad (19)$$

and from Gaussian mixture equations, the density of check node in iteration l has the following mean and variance values

$$m_u^{(l)} = \sum_{i=2}^{D_c} \rho_i m_{u,i}^{(l)}, \quad (20)$$

$$\sigma_u^{2(l)} = \sum_{i=2}^{D_c} \rho_i \left[\sigma_{u,i}^{2(l)} + (m_{u,i}^{(l)})^2 \right] - (m_u^{(l)})^2. \quad (21)$$

Therefore, the DE can be found by solving (19) for a check node with degree i and the distribution of a check node is then found using (20) and (21), iteratively.

C. Numerical Results of the Density Evolution Analysis

We solve the density evolution equations iteratively for a $(3, 6)$ regular LDPC code considering $m_u^{(0)} = 0$ and $\sigma_u^{2(0)} = 0$ as initial conditions. This provides the mean and the variance of check nodes' outputs and allows for the computation of the threshold for the given variances of internal decoder noise and communication channel noise.

Table I shows the relation between the threshold and the internal decoder noise variance σ_d^2 resulting from (17). It can be observed that the SNR threshold $\text{SNR}_{\text{th}} \triangleq (\frac{E_b}{N_0})_{\text{th}}$ increases as the internal decoder noise variance increases. This is in line with a similar observation in [15] on the performance of bit-flipping LDPC decoding in the presence of noisy message-passing over BSC channels, where the performance deteriorates as the cross-over probability of the internal decoder noise increases.

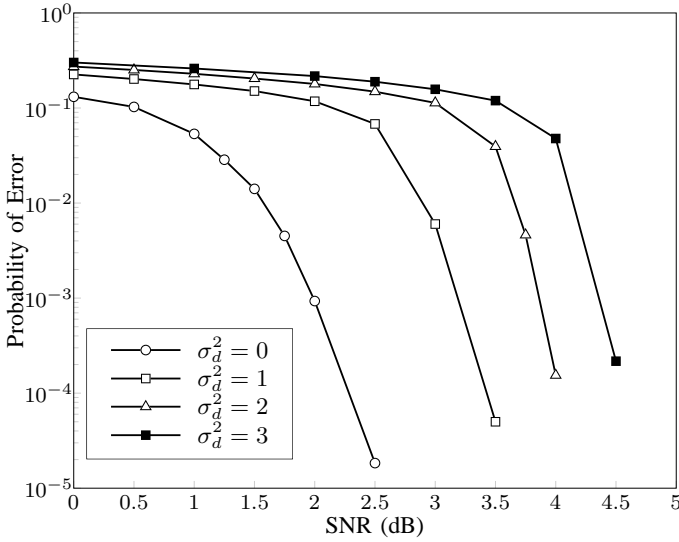


Fig. 4. Error probability performance of (3,6) regular finite length code with decoder noise variance σ_d^2 .

Simulation results confirm that our analytical results accurately predict the performance of finite-length codes as well. Fig. 4 depicts the simulation results for the performance of a finite-length (3,6) regular code with length $N = 1008$. It is evident that the threshold of this finite-length code is fairly the same as our analytical threshold for various values of the internal decoder noise variance σ_d^2 . One can use (20) and (21) to also track the density of irregular LDPC codes for noisy decoder.

In general, the presented analysis could be used to design irregular LDPC codes for noisy decoders. Since the problem of designing irregular LDPC codes by density evolution is not a convex problem, finding a good degree distribution requires complex computations and extensive search [27]. Therefore, in the remaining sections after investigating the performance limits of the noisy decoder by means of EXIT chart, we will introduce a simple and effective method to design robust LDPC codes for the noisy decoder.

IV. EXIT CHART ANALYSIS OF NOISY DECODER

The EXIT chart analysis, first introduced in the pioneering work of ten Brink [28], is a powerful tool for analyzing the performance of iterative turbo techniques. It is mainly based on keeping track of the mutual information of channel input bits and variable node and check node outputs.

Let X be a binary random variable denoting the BPSK modulated AWGN communication channel input which takes ± 1 values with equal probabilities. If $f(y)$ is the probability density function (pdf) of the communication channel soft output Y , then, the mutual information of X and Y for a symmetric channel [29], [30] is

$$I(X; Y) = \frac{1}{2} \sum_{x=\pm 1} \int_{-\infty}^{\infty} f(y|x) \log \left(\frac{f(y|x)}{f(y)} \right) dy. \quad (22)$$

In order to find the EXIT function of a noiseless decoder, the variable node and the check node EXIT functions can

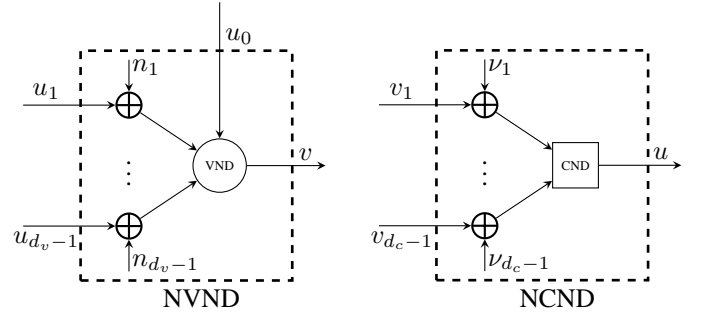


Fig. 5. Modification of decoder components (VND or CND) to noisy decoder components (NVND or NCND).

Algorithm 1 EXIT curve for NVND

- 1: **Input:** $\mathbf{I}_A, d_v, \sigma_n^2, \sigma_d^2, N$
- 2: **Output:** \mathbf{I}_E
- 3: $\mathbf{X} = \mathbf{1}$, [length N BPSK modulated codeword]
- 4: **for** $k = 1 : N$ **do** [compute decoder LLR inputs]

$$\mathbf{Y}(k) = \frac{2}{\sigma_n^2} (1 + n_c), n_c \sim \mathcal{N}(0, \sigma_n^2)$$

- 5: **end for**
- 6: **for** $i = 1 : \text{length}(\mathbf{I}_A)$ **do**

$$\sigma_A = J^{-1}(\mathbf{I}_A(i)),$$

where

$$J(\sigma) \triangleq 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\sigma^2/2)^2}{2\sigma^2}} \log_2(1 + e^{-x}) dx$$

- 7: **for** $j = 1 : N$ **do** [compute a priori LLRs]

$$\mathbf{AP}(j) = \frac{\sigma_A^2}{2} + n, n \sim \mathcal{N}(0, \sigma_A^2)$$

- 8: **end for**
- 9: [compute noisy variable node outputs using (5)]

$$\mathbf{E} = \text{NVND}(d_v, \mathbf{AP}, \mathbf{Y}, \sigma_d^2)$$

- 10: [Compute extrinsic mutual information using (22)]

$$\mathbf{I}_E(i) = I(\mathbf{X}, \mathbf{E})$$

- 11: **end for**
-

be computed using a J -function [8], which directly results from the consistency assumption of the decoder. Since this assumption is violated in the case of a noisy decoder, to compute the a priori and extrinsic mutual information, we compute these values according to the definition of mutual information.

To obtain the EXIT curves for the noisy LDPC decoder, we use **Algorithm 1** and the empirical distributions computed by running simulations for each degree of variable node and check node. Specifically for each decoder component, we compute the extrinsic mutual information I_E corresponding to a priori mutual information I_A for the noisy variable (check) node decoder. To maintain the desired structure of the iterative decoder [8] for the noisy decoder, we modified the variable nodes and check nodes as shown in Fig. 5. In fact, we have

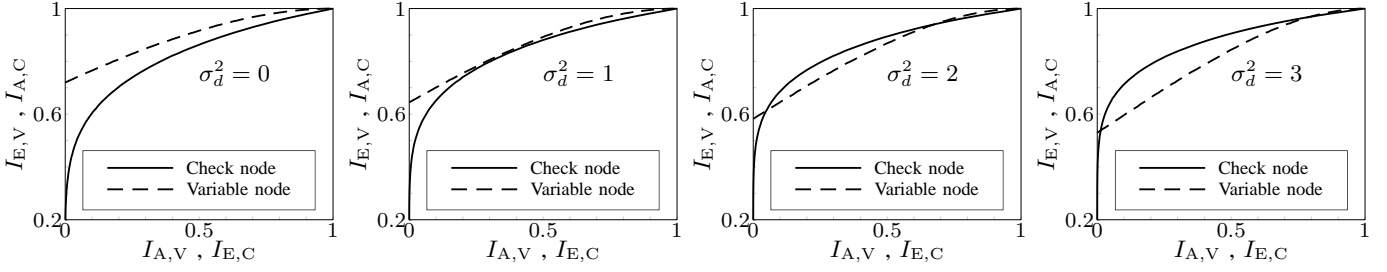


Fig. 6. EXIT charts of (3,6) regular LDPC code resulting from **Algorithm 1** for SNR= 3 dB and different decoder noise variances σ_d^2 .

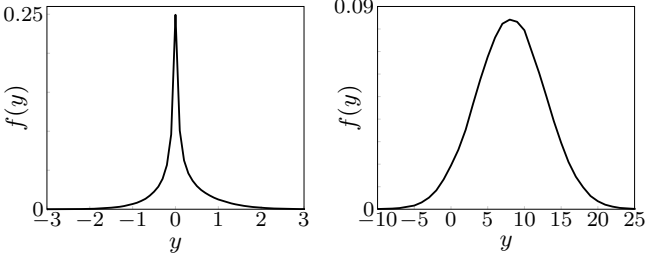


Fig. 7. Empirical distributions of outputs of check (left) and variable (right) nodes for $I_A = 0.5$, $\sigma_d^2 = 1$ and SNR = 3 dB.

considered decoder noise as a part of variable (check) nodes and call them noisy variable (check) node decoders, NVND (NCND).

In **Algorithm 1**, the proposed method for finding the EXIT curve of NVND is described. The procedure of finding EXIT curves for the noisy check node is similar to that of the noisy variable node; however, for the check node the communication channel output is not fed to the check node, i.e., the only input of the noisy check node is the vector of a priori LLRs from variable nodes (**AP**). The presented algorithm is an extension of the algorithm 7.4 in [31] for Turbo codes.

It is noteworthy that for a noiseless decoder, input messages to each decoder component (VND or CND) are modeled by Gaussian random variables of mean μ_A and variance $\sigma_A^2 = 2\mu_A$. Then the mutual information I_A of each input message and the channel input X (a priori mutual information) can be computed using

$$I_A = J(\sigma_A).$$

This relation implies a one-to-one mapping from each I_A to a σ_A (and consequently μ_A). However, for a noisy decoder there is not such a one-to-one mapping and for each I_A different sets of (μ_A, σ_A^2) may be found. One way to tackle this problem is to assume consistency for input messages to the noisy components (NVND and NCND) as stated in step 6 of **Algorithm 1**, and calculate each a priori LLR as in step 7. Then, the input messages to VND and CND are no longer consistent (since they are corrupted with decoder noise). Since we simulate each decoder component separately, the assumption made does not affect the computation of EXIT curves for the other components, and as our experiments validate, the suggested approach provides very accurate EXIT curves.

Fig. 6 illustrates the evolution of EXIT charts of (3,6) regular LDPC code for different values of decoder noise variance σ_d^2 and for communication channel SNR 3 dB. For

$\sigma_d^2 = 0$, there is an open tunnel between the curves, and increasing σ_d^2 to one makes the tunnel tighter. However, for $\sigma_d^2 = 2$ and $\sigma_d^2 = 3$ the variable and the check EXIT curves cross each other. The EXIT charts in Fig. 6 illustrate that the SNR threshold of (3,6) regular LDPC code is less than 3 dB when $\sigma_d^2 = 0$ and $\sigma_d^2 = 1$ and is greater than 3 dB when $\sigma_d^2 = 2$ and $\sigma_d^2 = 3$. This is in line with the results obtained from the density evolution analysis, as shown in Table I.

Fig. 7 depicts the empirical distributions $f(y)$ for the outputs of NVND of degree $d_v = 3$ and NCND of degree $d_c = 6$, used in (22) to find the extrinsic mutual information I_E . We observe that the distribution of the nodes' outputs are light-tailed for the check node and normal-tailed for the variable node. This motivates the use of numerical integration in (22) by integrating over a limited integral domain.

In the following section, we will use the EXIT curves resulting from **Algorithm 1** to design robust LDPC codes for noisy decoders.

V. DESIGN OF ROBUST IRREGULAR LDPC CODES FOR NOISY DECODER

In this section, our goal is to design robust irregular LDPC codes for noisy decoder. The design procedure of LDPC codes is based on fitting EXIT curves corresponding to given variable and check degree distributions. In [8], for a noiseless decoder, right-regular LDPC codes are designed for a fixed check node degree of d_c , by fitting a weighted sum of the EXIT curves of variable nodes to the check node EXIT curve. In this work, we design irregular LDPC codes and allow more than one degree for both variable nodes and check nodes. Our benchmark for comparisons are irregular LDPC codes with the same rates designed for a noiseless decoder in [29]. We verify the performance of the designed codes by the simulations of finite-length LDPC codes.

It is noteworthy that when the exchanged messages in the decoder are not consistent, the weighted sum of the EXIT chart curves of variable node and check node are not the exact curves of the irregular code. However, our simulation results indicate that this approach is accurate enough for the case with a noisy decoder.

A. Code Design Algorithm

Using the EXIT curves obtained from **Algorithm 1**, we propose a simple method for the design of irregular codes for the noisy decoder. Similar to the code design approach

Algorithm 2 Robust LDPC code design

```

1: Input:  $d_c, D_v, r, \sigma_d^2, \Delta, \mathbf{S}$ 
2: Output: code  $\mathcal{C}^* = (\rho(x)^*, \lambda(x)^*)$ 
3: initialize  $\text{SNR}_{\text{th}}$ 
4:  $\sigma_n^2 = (2r\text{SNR}_{\text{th}})^{-1}$ 
5: for  $i = 1 : M + 1$  do
6:    $\alpha = \mathbf{S}(i)$ 
7:   [compute EXIT curves of check nodes with degree
8:    $d_c - 1$  and degree  $d_c$  using Algorithm 1]
9:   [compute effective EXIT curve  $I_{A,\text{NCND}}$  using  $\rho(x)$ 
10:   in (23)]
11:   [compute EXIT curves of variable nodes with degrees
12:    $d_v = 2$  to  $d_v = D_v$  using Algorithm 1]
13:   [check if there is a  $\lambda(x)$  such that  $\mathcal{C} = (\rho(x), \lambda(x))$ 
14:   satisfies the constraints in (25)]
15:   if code  $\mathcal{C}$  exists then
16:      $\text{SNR}_{\text{th}} = \text{SNR}_{\text{th}} - \Delta$ 
17:      $\mathcal{C}^* = \mathcal{C}$ 
18:     go to step 4
19:   end if
20: end for

```

proposed in [29], we restrict our attention to irregular codes with two consecutive check node degrees as

$$\rho(x) = \alpha x^{d_c-1} + (1 - \alpha)x^{d_c}, \quad (23)$$

and variable node degrees as

$$\lambda(x) = \sum_{i=2}^{D_v} \lambda_i x^{i-1}. \quad (24)$$

The *effective* EXIT curve of a noisy irregular check (variable) node $I_{A,\text{NCND}}$ ($I_{E,\text{NVND}}$) is obtained by averaging over the EXIT curves of check (variable) nodes with constituent check (node) degrees [9].

In this paper, we refer to a candidate code $\mathcal{C} = (\rho(x), \lambda(x))$ of rate r as a *successful code* for a given SNR, if it satisfies the following constraints

$$\begin{aligned}
i) \quad & \lambda(1) = 1, \\
ii) \quad & \rho(1) = 1, \\
iii) \quad & r = 1 - \frac{\int_0^1 \rho(x) dx}{\int_0^1 \lambda(x) dx}, \\
iv) \quad & I_{E,\text{NVND}} \succ I_{A,\text{NCND}},
\end{aligned} \quad (25)$$

where the last constraint implies that the effective EXIT curve of its variable node lies above the effective EXIT curve of its check node.

For a given code rate r , we are looking for the code $\mathcal{C}^* = (\rho(x)^*, \lambda(x)^*)$ corresponding to the threshold SNR, SNR_{th} , i.e., the minimum SNR for which there is a successful code. In **Algorithm 2**, we set the check node degree d_c , the maximum variable node degree D_v , and design rate r . In order to speed up the design procedure, we let α take limited values in $\mathbf{S} = [0 : 1/M : 1]$. Then, for a given SNR_{th} , by varying α from zero to one, we check if there is a variable degree distribution $\lambda(x)$, as in (24), such that the constraints in (25)

are satisfied. One way to do this is for each α to find $I_{A,\text{NCND}}$ according to (23) and check if there is a vector $(\lambda_2, \dots, \lambda_{D_v})$ for which the constraints in (25) are feasible. If so, we form the code $\mathcal{C} = (\rho(x), \lambda(x))$. Subsequently, we reduce SNR_{th} by Δ and the algorithm does another iteration; otherwise, it terminates and the successful code from the previous iteration is selected.

The EXIT curves of a check node do not depend on SNR_{th} ; By changing SNR_{th} , the EXIT curves of variable nodes, and consequently $I_{E,\text{NVND}}$ are affected. Using this fact, the complexity of the proposed algorithm may be further reduced by pre-computing $I_{A,\text{NCND}}$ for each value of α once and storing them before running the algorithm. In the following section, we will present some results illustrating the application of the proposed algorithm in the design of robust LDPC codes for noisy decoders.

B. Design Examples

From Table 1 of [29], for the maximum variable degree of four, the introduced code of rate one-half has two types of check nodes with degrees five and six. The threshold of this code is 0.8085 dB and has the following degree distributions

$$\begin{aligned}
\lambda(x) &= 0.384x + 0.042x^2 + 0.574x^3, \\
\rho(x) &= 0.241x^4 + 0.759x^5.
\end{aligned}$$

However, when the decoder is not perfect, the threshold and also the performance of this code degrade. For instance, in noisy decoder the threshold has increased by about 1.7 dB and 2.5 dB for decoder noise variances $\sigma_d^2 = 0.5$ and $\sigma_d^2 = 1$, respectively.

Considering the same constraints in check degree distribution and maximum variable degree, we have designed an irregular one-half rate LDPC code for the noisy decoder with $\sigma_d^2 = 0.5$. Using **Algorithm 2**, we obtained *code A* with the following degree distributions

$$\begin{aligned}
\lambda(x) &= 0.453x + 0.547x^3, \\
\rho(x) &= 0.451x^4 + 0.549x^5.
\end{aligned}$$

Fig. 8 shows the performance of *Code A* and the designed code in [29] for different decoder noise variances. When $\sigma_d^2 = 0.5$ (solid curves), *Code A* provides a better performance compared to the code in [29]. By increasing the decoder noise variance to $\sigma_d^2 = 0.7$ (dashed curves) *Code A* still outperforms the code in [29], while by decreasing the decoder noise variance to $\sigma_d^2 = 0.3$ (dotted curves) the codes show almost the same performance. It is evident that *Code A* performs well when the decoder noise power varies in the proximity of the designed variance. It is noteworthy that in our simulations the Tanner graph of codes are free from cycles of length four. The maximum number of decoder iterations is set to 80, and for each SNR, the error probability is computed after a total number of 50 block errors occur.

As another example of code design, considering the same constraints on degree distributions, we have designed an irregular code (*Code B*) for noisy decoder with $\sigma_d^2 = 1$. The

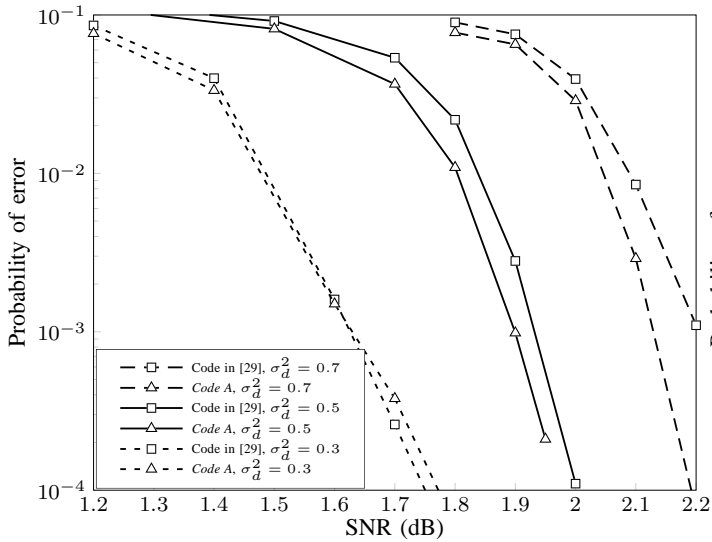


Fig. 8. Error probability performances of *Code A* and the code in [29] for different decoder noise variances, $N = 10^4$.

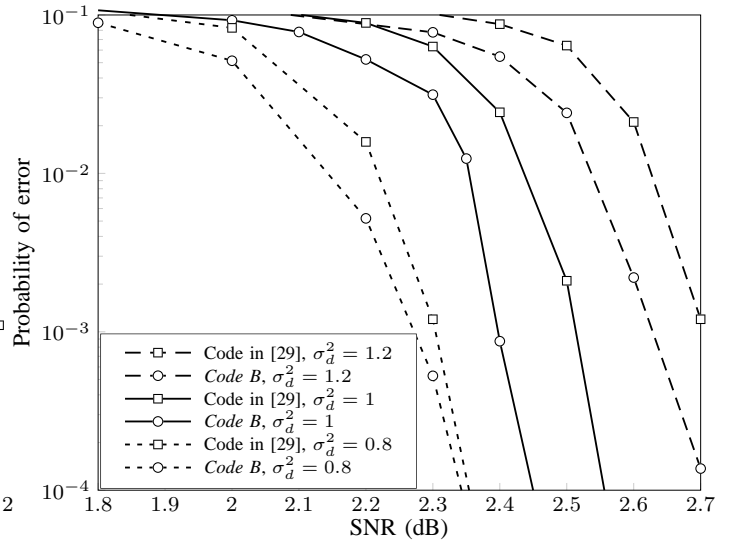


Fig. 9. Error probability performances of *Code B* and the code in [29] for different decoder noise variances, $N = 10^4$.

degree distributions of this code are

$$\begin{aligned}\lambda(x) &= 0.4808x + 0.5192x^3, \\ \rho(x) &= 0.553x^4 + 0.447x^5.\end{aligned}$$

Fig. 9 shows the simulation results of *Code B* and the designed code in [29] for different decoder noise variances. When $\sigma_d^2 = 1$ (solid curves), *Code B* shows better performance compared to the code in [29]. Similar observations are made when the decoder noise variance (dashed curves) increases to 1.2 or decreases to 0.8 (dotted curves).

In the above designs, we considered two consecutive check degrees for a fair comparison with the code in [29]. However, the presented approach may be used to design irregular LDPC codes with arbitrary check degrees as

$$\rho = \sum_{i=2}^{D_c} \rho_i x^{i-1}.$$

The constraints in (25) remain linear and hence the same procedure still applies. We emphasize that the range of decoder internal noise power that we considered here (between 0 and 1.2) is rather conservative. For a scalar quantization of a Gaussian random variable, a single bit change in the quantization bitrate scales the quantization noise variance by a factor of about 3.4 [32]. According to our simulation results, the proposed robust LDPC code design indicates a higher gain when the decoder internal noise power is stronger.

VI. CONCLUSIONS

We considered a noisy message-passing scheme for the decoding of LDPC codes over AWGN communication channels. We modeled the internal decoder noise as additive white Gaussian and observed the inconsistency of the exchanged message densities in the iterative decoder. For the inconsistent LDPC decoder, a density evolution scheme was formulated to track both the mean and the variance of the messages exchanged in the decoder. We quantified the increase of the

decoding threshold SNR as a consequence of the internal decoder noise based on a density evolution analysis. We also analyzed the performance of the noisy decoder using EXIT charts. We introduced an algorithm based on the computed EXIT charts to design robust irregular LDPC codes. The designed codes partially compensate the performance loss due to the decoder internal noise. In this work, we modeled the decoder noise as AWGN on the exchanged messages, however, an interesting future step is to incorporate other noise models possibly directly obtained from practical decoder implementations. One may also use this research to design codes that are robust to decoder internal noise whose power may vary in a given range depending on possible types of implementation.

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